

# ON THE DEFORMATION OF ABELIAN INTEGRALS.

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**ABSTRACT.** We consider the deformation of abelian integrals which arose from the study of SG form factors. Besides the known properties they are shown to satisfy Riemann bilinear identity. The deformation of intersection number of cycles on hyperelliptic curve is introduced.

The subject of this paper is related to the problem of description of all local fields in SG theory the solution to which will be published elsewhere. However, the result presented in this paper which is the generalization of Riemann bilinear identity to deformed abelian integrals constitutes rather isolated and, presumably, the most mathematically interesting part of the problem, for that reason the author decided to publish it separately.

The integrals which provide a deformation of periods of hyperelliptic differentials were introduced several years ago for the description of SG form factors [1], let us explain how they are constructed. Consider the function

$$\varphi(\alpha) = C \exp \left( -2 \int_0^\infty \frac{\sin^2 \frac{k\alpha}{2} \operatorname{sh} \frac{\pi+\xi}{2} k}{k \operatorname{sh} \frac{\xi k}{2} \operatorname{sh} \pi k} dk \right)$$

where  $C$  is certain constant [1] which is needed for  $\varphi$  to satisfy the relation

$$\varphi(\alpha + \pi i) \varphi(\alpha) = \frac{1}{4 \operatorname{sh} \frac{\pi}{\xi} (\alpha + \frac{\pi i}{2}) \operatorname{ch}(\alpha)} \quad (1)$$

without any additional multiplier in RHS. Other important properties of  $\varphi$  are

$$\varphi(\alpha + i\xi) = \varphi(\beta) \frac{\operatorname{ch} \frac{1}{2} (\alpha - \frac{\pi i}{2})}{\operatorname{ch} \frac{1}{2} (\alpha + \frac{\pi i}{2} + i\xi)}, \quad \varphi(\alpha + 2\pi i) = -\varphi(\alpha) \frac{\operatorname{sh} \frac{\pi}{\xi} (\alpha + \frac{\pi i}{2})}{\operatorname{sh} \frac{\pi}{\xi} (\alpha + \frac{3\pi i}{2})} \quad (2)$$

The function  $\varphi(\alpha)$  does not have other singularities for  $0 \leq \operatorname{Im} \alpha \leq 2\pi$  but the simple poles at the points  $\alpha = \frac{\pi i}{2} + i\xi k$ ,  $k \geq 0$  (how many of them are in the

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strip depends on the value of  $\xi$ ). Asymptotically when  $\alpha \rightarrow \pm\infty$  the function  $\varphi(\alpha)$  behaves as

$$\varphi(\alpha) \sim \exp\left(-\frac{1}{2}\left(\frac{\pi}{\xi} + 1\right)|\alpha|\right)$$

The deformation of hyperelliptic abelian integral is defined as follows [1,2]. Consider  $2n$  points  $\beta_1, \dots, \beta_{2n}$  (analogues of branching points). Then for two given polynomials  $Q(a)$  and  $L(A)$  (which can also depend respectively on  $b_j = e^{\frac{2\pi}{\xi}\beta_j}$  and on  $B_j = e^{\beta_j}$  as on parameters) we consider the pairing  $\langle Q(a), L(A) \rangle$  defined by an integral:

$$\langle Q(a), L(A) \rangle \equiv \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) Q(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha \quad (3)$$

where for further convenience we introduced the notation:

$$\tilde{\varphi}(\alpha, \beta) = \varphi(\alpha - \beta) \exp\left(-\frac{1}{2}\left(\frac{\pi}{\xi} + 1\right)(\alpha + \beta)\right)$$

The integral is convergent for  $1 \leq \deg L(A) \leq 2n - 1$  and  $\deg Q(a) \leq n - 1$ . However, as it is explained in [1,2] the last condition can be relaxed, with a proper regularization the integral can be defined for arbitrary polynomial  $Q(a)$  in such a way that one can deal with the deformed integral as with usual one: deform contours for example. For the sake of completeness we shall describe the regularization in the Appendix. Moreover, only  $2n - 1$  of these polynomials give really different integrals, we shall comment on this point later. In what follows we shall use equally often the variables

$$\alpha, \beta_j; \quad a = e^{\frac{2\pi}{\xi}\alpha}, b_j = e^{\frac{2\pi}{\xi}\beta_j}; \quad A = e^\alpha, B_j = e^{\beta_j}$$

Let us explain the relation to hyperelliptic integrals. Consider the limit when  $\xi \rightarrow \infty$  but the variables  $b_j$  remain finite, which means that  $\beta_j$  are getting simultaneously rescaled. Let us require that  $b_1 < b_2 < \dots < b_{2n}$ . Then it can be shown that the following asymptotic formula holds

$$\langle a^p, A^k \rangle \sim \prod_{j=1}^k B_j \int_{\gamma_k} \frac{a^p}{\sqrt{P(a)}} da \quad (4)$$

where the polynomial  $P(a)$  is given by  $P(a) = \prod (a - b_j)$ , the cycles  $\gamma_j$  are those drawn around the branching points  $b_i$  and  $b_{i+1}$  on the hyperelliptic surface  $c^2 = P(a)$ . In the papers [3,4] the formula (4) was proven in a little different context: the limit  $\xi \rightarrow \infty$  was taken first, and the  $\beta_j$  were rescaled.

The reason the asymptotical formula (4) to exist is hidden in the properties of the function  $\tilde{\varphi}(\alpha, \beta)$  for  $\xi \rightarrow \infty$ :

$$\tilde{\varphi}(\alpha, \beta) \sim (A^{-1}\theta(\alpha - \beta) + B^{-1}\theta(\beta - \alpha)) \frac{1}{\sqrt{(a - b_j)}},$$

the equation (1) and the second equation from (2) turn into

$$\left( \frac{1}{\sqrt{(a - b_j)}} \right)^2 = \frac{1}{(a - b_j)}, \quad \frac{d}{da} \frac{1}{\sqrt{(a - b_j)}} = -\frac{1}{2} \frac{1}{(a - b_j)} \frac{1}{\sqrt{(a - b_j)}}$$

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Let us emphasize the most interesting property of this deformation of hyperelliptic integrals: the integration over different cycles is replaced by the integration over all the real axis with additional weights  $A^k = e^{k\alpha}$  inserted into the integral. In what follows we shall call the monomial  $A^k$  a deformation of  $\gamma_k$ . It is natural to consider usual cycles as generators of an exterior algebra over the ring of integer numbers, the deformed cycles constitute a natural exterior algebra over the ring of quasiconstants: symmetric Laurent polynomials in variables  $B_j$ .

Let us list the properties of the hyperelliptic differentials. We do not expect the Riemann surface itself to allow a deformation, only the globally defined objects: cycles and differentials with their periods. That is why we are interested in the differentials of the first and of the second kind, i.e. in those without simple poles because including the differentials with the simple poles is the same as considering points on the surface. It is convenient for our goals to consider the following basis of the differentials of the first and of the second kind respectively

$$\begin{aligned}\eta_p &= \frac{a^{p-1}}{\sqrt{P(a)}} da, \\ \zeta_p &= \frac{1}{\sqrt{P(a)}} \left( \sum_{k=n}^{2n-p} + \frac{1}{2} \sum_{k=1}^{n-1} \right) ((-1)^{k+p} (k-p) a^{k-1} \sigma_{2n-p-k}(b_1, \dots, b_{2n}) da\end{aligned}\tag{5}$$

where  $p = 1, \dots, n-1$  (recall that the genus of the surface is  $n-1$ ),  $\sigma_i(b_1, \dots, b_{2n})$  are the elementary symmetric polynomials. We are missing the differential with the polynomial in front of  $\frac{1}{\sqrt{P(a)}}$  whose leading power is  $n-1$  in (5) because it is a differential with the simple pole at the infinity. The differentials with the polynomials of higher degree than  $2n-2$  can be reduced to those with lower degree by subtracting the exact forms  $d(a^k \sqrt{P(a)})$ . We are going to generalize to the deformed case the following properties of the differentials and cycles:

**1. Total derivatives.** The differentials are defined up to the adding of an exact form.

**2. The independent cycles.** There are only  $2n-2$  independent cycles, there is one relation of linear dependence between the cycles  $\gamma_i$ , namely,  $\sum \gamma_{2i-1} \sim 0$  which means that for any differential without residues on the surface one has

$$\sum_{i=1}^n \int_{\gamma_{2i-1}} \omega = 0\tag{6}$$

**3. Riemann bilinear identity.** For any two differentials without residues one has

$$\sum_{i=1}^{n-1} \left( \int_{\alpha_i} \omega_1 \int_{\beta_i} \omega_2 - \int_{\alpha_i} \omega_2 \int_{\beta_i} \omega_1 \right) = \sum_{\text{poles}} \text{res}(\Omega_1 \omega_2) \equiv \omega_1 \circ \omega_2$$

where  $\Omega_1$  is a primitive function for  $\omega_1$ ;  $\alpha_i, \beta_i$  is a canonic basis with the intersection numbers:

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}$$

It can be chosen as  $\alpha_i = \gamma_{2i}$ ,  $\beta_i = \sum_{k=1}^i \gamma_{2k-1}$ . The usual way of proving of the relation is to integrate the 1-form  $\Omega_1 \omega_2$  over the canonic polygon associated to the

Riemann surface. This way is not easy to generalize to the deformed case, so we shall rely on an alternative possibility which is described later.

The differentials (5) are constructed in order that they constitute a canonic basis of differentials:

$$\eta_i \circ \eta_j = \zeta_i \circ \zeta_j = 0, \quad \eta_i \circ \zeta_j = \delta_{ij}$$

This property of the differentials allows to write a dual form of the Riemann bilinear identity (equivalent to the original one):

$$\sum_{i=1}^{n-1} \left( \int_{\delta_1} \eta_i \int_{\delta_2} \zeta_i - \int_{\delta_2} \eta_i \int_{\delta_1} \zeta_i \right) = \delta_1 \circ \delta_2 \quad (7)$$

for any two cycles  $\delta_1, \delta_2$  on the surface. This is the formulation which allows easier generalization for deformed case, so let us sketch the proof of (7).

Consider the identity which follows from a simple algebra:

$$\sum_{i=1}^{n-1} (\eta_i(a) \zeta_i(a') - \zeta_i(a) \eta_i(a')) = \left( \frac{\partial}{\partial a} \frac{\sqrt{P(a)}}{(a-a')\sqrt{P(a')}} - \frac{\partial}{\partial a'} \frac{\sqrt{P(a')}}{(a'-a)\sqrt{P(a)}} \right) da da'$$

Integrate this identity over  $\delta_1 \times \delta_2$ . In the LHS one gets the LHS of (7) while in the RHS one has the integral of total derivatives which is evidently sitting on the contact terms, the calculation of the latter gives the intersection number of the cycles.

Let us explain how to deform all these properties of the abelian integrals. The generalization of the first and the second properties are well known [2,3], but we are going to discuss them for the sake of completeness.

**1. The deformation of the total derivative.** For the given polynomial  $Q(a)$  let us construct the polynomials of the form

$$\widehat{Q}(a) = \left( \prod_{j=1}^{2n} (a\tau - b_j) Q(a\tau^4) - \prod_{j=1}^{2n} (a\tau^{-1} - b_j) Q(a) \right) a^{-1}$$

here and later on  $\tau = \exp(\frac{\pi^2 i}{\xi})$ . We claim that substituting  $\widehat{Q}(a)$  into the integral (3) one gets zero. Formally it goes as follows:

$$\begin{aligned} \langle \widehat{Q}(a), L(A) \rangle &= \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) \widehat{Q}(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha = \\ &= \left\{ \int_{-\infty+2\pi i}^{\infty+2\pi i} - \int_{-\infty}^{\infty} \right\} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) (e^{\frac{2\pi}{\xi}\alpha} \tau^{-1} - e^{\frac{2\pi}{\xi}\beta_j}) Q(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) d\alpha = 0 \end{aligned} \quad (8)$$

We used the properties of the function  $\varphi(\alpha)$  to write this identity. The latter integral is zero because the integrand does not have singularities in the strip  $0 < \text{Im}\alpha < 2\pi$ . In fact, both integrals in (8) can be divergent, but the manipulations with them can be justified with the regularization described at the Appendix. Notice that in the limit  $\xi \rightarrow \infty$  the polynomial  $\widehat{Q}(a)$  turns into

$$\frac{2\pi}{\xi} \left( \frac{d}{da} Q(a) - \frac{1}{2} \frac{d}{da} P(a) \right)$$

as it has to be for the total derivative.

**2. Independent deformed cycles.** Using the properties of the function  $\varphi(\alpha)$  one can write an identity

$$\begin{aligned} & \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) \left\{ \prod_{j=1}^{2n} (e^{\alpha} + ie^{\beta_j}) - \prod_{j=1}^{2n} (e^{\alpha} - ie^{\beta_j}) \right\} Q(e^{\frac{2\pi}{\xi}\alpha}) e^{\frac{2\pi}{\xi}\alpha} d\alpha = \\ & = \left\{ \int_{-\infty+i\xi}^{\infty+i\xi} - \int_{-\infty}^{\infty} \right\} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) (e^{\alpha} - ie^{\beta_j}) Q(e^{\frac{2\pi}{\xi}\alpha}) e^{\frac{2\pi}{\xi}\alpha} d\alpha \end{aligned} \quad (9)$$

Formally, the integral in the RHS is zero (the integrand is regular) unless we pick up the contributions from the segments  $(-\infty, -\infty + i\xi)$  and  $(\infty, \infty + i\xi)$ . The latter happens for the analogues of the third kind differential with a simple pole at infinity. Again, one has to be careful with possible divergences and to make the regularization (see Appendix). For the analogues of the differentials of the first and of the second kind one shows that the integral is zero, hence, decomposing the LHS of (9) one has:

$$\sum_{k=0}^{n-1} \sigma_{2n-2k-1}(B_1, \dots, B_{2n}) \langle Q(a), A^{2k+1} \rangle = 0$$

This identity is analogous to (6), the only new point being that the deformed cycles are linear dependent over the ring of symmetric functions of  $B_j$  (quasiconstants).

It is wonderful that the properties 1 and 2 being of very different nature in underformed case allow quite similar proofs after the deformation. That means that cycles and differentials become much more similar after the deformation.

**The deformation of Riemann bilinear identity.** Let us introduce the basis of deformed first and second kind differentials similar to (5). The corresponding polynomials are given by

$$\begin{aligned} R_p(a) &= a^{p-1}, \\ S_p(a) &= \left( \sum_{k=n}^{2n-p} + \frac{1}{2} \sum_{k=1}^{n-1} \right) ((-1)^{k+p} (\tau^{k-p} - \tau^{p-k}) a^{k-1} \sigma_{2n-p-k}(b_1, \dots, b_{2n})) \end{aligned}$$

Now, consider the simple algebraic identity

$$\sum_{i=1}^{n-1} (R_p(a) S_p(a') - S_p(a) R_p(a')) = X(a, a') - X(a', a)$$

where

$$X(a, a') = a^{-1} \tau^{-1} \frac{\prod (a\tau - b_j)}{a\tau - a'\tau^{-1}} - a^{-1} \tau \frac{\prod (a\tau^{-1} - b_j)}{a\tau^{-1} - a'\tau}$$

Now let us multiply this identity by

$$\prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) \tilde{\varphi}(\alpha', \beta_j) L(e^{\alpha}) M(e^{\alpha'}) e^{\frac{2\pi}{\xi}(\alpha + \alpha')}$$

and integrate the result over  $\alpha$  and  $\alpha'$ . The nontrivial thing is hidden in the integration of the RHS. Indeed, consider the first part of the corresponding integral:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) \tilde{\varphi}(\alpha', \beta_j) L(e^\alpha) M(e^{\alpha'}) X(e^{\frac{2\pi\alpha}{\xi}}, e^{\frac{2\pi\alpha'}{\xi}}) e^{\frac{2\pi}{\xi}(\alpha+\alpha')} d\alpha d\alpha' = \\
& = \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha', \beta_j) e^{\frac{2\pi}{\xi}\alpha'} M(e^{\alpha'}) \\
& \left\{ \int_{-\infty+2\pi i}^{\infty+2\pi i} - \int_{-\infty}^{\infty} \right\} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) (e^{\frac{2\pi}{\xi}\alpha} \tau^{-1} - e^{\frac{2\pi}{\xi}\beta_j}) \frac{e^{-\frac{\pi}{\xi}(\alpha+\alpha')}}{\text{sh} \frac{\pi}{\xi}(\alpha - \alpha' - \pi i)} L(e^\alpha) d\alpha d\alpha'
\end{aligned} \tag{10}$$

The last integral over  $\alpha$  is easy to take since it is sitting on the simple poles of  $1/\text{sh} \frac{\pi}{\xi}(\alpha - \alpha' - \pi i)$ . The pole at  $\alpha = \alpha' + \pi i$  always exists. For  $\xi < \pi$  the additional poles at  $\alpha = \alpha' + \pi i \pm i\xi m$  occur but the contributions from them can be shown to cancel each others. So, the only contribution comes from the first pole. With the help of (1) the integral (10) is shown to be expressed in terms of quasiconstants only:

$$\int_0^{\infty} \prod_{j=1}^{2n} \frac{M(A)L(-A)}{A^2 + B_j^2} A^{-1} dA$$

Gathering all pieces we get the deformed version of the Riemann bilinear identity:

$$\sum_{p=1}^{n-1} (\langle R_p(a), L(A) \rangle \langle S_p(a), M(A) \rangle - \langle S_p(a), L(A) \rangle \langle R_p(a), M(A) \rangle) = L \circ M$$

where the intersection number for the polynomials  $L$  and  $M$  is defined as

$$L \circ M = \int_{-\infty}^{\infty} \frac{L(A)M(-A) - L(-A)M(A)}{\prod (A^2 + B_j^2)} A^{-1} dA$$

Obviously, the only nonzero intersections are those of even degrees of  $A$  with odd ones. So, the deformed analog of canonic basis can be constructed in such a way that  $A^{2k}$  are the analogues of  $a$ -cycles while the  $b$ -cycles are taken as suitable linear combinations (with quasiconstant coefficients) of  $A^{2k-1}$ . In terms of the canonic basis one can write down the deformed version of the original form of Riemann bilinear identity.

To finish this paper let us make several comments. First, as it has been already said, the results obtained here are needed for the solution of an important physical problem, namely, the problem of description of all local fields in SG theory in terms of the form factors bootstrap. The duality between fields and particles in SG theory can be described as the duality between deformed cycles and differentials described here. So, this paper is not just a mathematical exercise. Another interesting point is a possibility of further deformation. It must be possible to introduce one more parameter of deformation in such a way that the duality between polynomials described here is replaced by the duality between algebraic expressions written in terms of Jacobi  $\theta$ -functions, the first step in this direction is done in [5].

## APPENDIX.

In this Appendix we explain the regularization of the integral (3) introduced in [1,2]. Consider the integral

$$\int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) Q(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha$$

which is defined for  $\deg(Q(a)) \leq n-1$  (the condition  $\deg(L(A)) \leq 2n-1$  is always implied). The definition of the regularized integral for arbitrary polynomial  $Q$  is given by

$$\begin{aligned} & \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) Q(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha \equiv \\ & \int_{-\infty + \frac{\pi i}{2} - i0}^{\infty + \frac{\pi i}{2} - i0} \prod_{j=1}^{2n} \frac{\tilde{\varphi}(\alpha, \beta_j)}{e^{\frac{2\pi}{\xi}\alpha} \tau^{-3} - e^{\frac{2\pi}{\xi}\beta_j}} Q_1(e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha + \\ & + \int_{\Gamma} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) Q_2(\tau^4 e^{\frac{2\pi}{\xi}\alpha}) L(e^\alpha) e^{\frac{2\pi}{\xi}\alpha} d\alpha \end{aligned} \quad (A)$$

where the contour  $\Gamma$  is drawn around the points  $\alpha = \beta_j - \frac{\pi i}{2} - i\xi k$  for  $k = 0, \dots, \left[\frac{\pi}{\xi}\right]$ , polynomials  $Q_1$  and  $Q_2$  are defined by the equation:

$$\begin{aligned} Q(a) \prod (a\tau^{-3} - b_j) &= Q_1(a) + \\ &+ \tau^{-4} Q_2(a) \prod (a\tau^{-1} - b_j) - Q_2(a\tau^4) \prod (a\tau^{-3} - b_j) \end{aligned}$$

It can be shown that the polynomials satisfying this relation and the additional requirement  $\deg(Q_1) \leq 3n-1$  (which is enough for integrals in (A) to converge) can be found for any  $Q$ , and that possible ambiguity in definition of  $Q_1, Q_2$  is irrelevant for the value of RHS of (A) [2].

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